

Fixed point theorem for reflexive Banach spaces and uniformly convex non positively curved metric spaces

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Abstract. This article generalizes the work of Ballmann and Świątkowski to the case of Reflexive Banach spaces and uniformly convex Busemann spaces, thus giving a new fixed point criterion for groups acting on simplicial complexes.

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1 Introduction

For a finite graph (V, E) the Laplacian is a positive operator defined on functions $f : V \rightarrow \mathbb{R}$. One can generalize the definition of the Laplacian for a simplicial complex X of any dimension and for such a complex the Laplacian is again a positive.

Ballmann and Świątkowski in [BŚ97] and independently Żuk's in [Żuk96] used the geometric information given by the Laplacian eigenvalues to give criteria for the vanishing of cohomologies of a group Γ acting on a simplicial complex. The most famous result of this type is the Żuk criterion which states that a group acting geometrically (i.e. cocompactly and proper discontinuously) on a 2-dimensional simplicial complex has property (T) if the smallest positive Laplacian eigenvalues at the link of every vertex is large enough. It is well known

that in the above setting, property (T) is equivalent to a fixed point property for action by isometries on a Real Hilbert space (see for instance [BdlHV08]).

In this article, we generalize the Żuk criterion to Reflexive Banach spaces and Uniformly convex non positively curved Busemann spaces and get a fixed point criterion for those spaces relaying on the geometry of the links of vertices. The method that we use is basically taken from Gromov in [Gro03] (3.11), but we improve it so it doesn't require any scaling limit arguments (and generalize the form of the energy function).

Structure of the paper. The first section gathers needed results about groups acting on simplicial complexes, Uniformly convex Busemann non positively curved spaces and Reflexive Banach spaces. The second section contains the main theorem and its proof.

2 Framework and Preliminaries

2.1 General Settings

Throughout this paper X is a simplicial complex of dimension $n \geq 2$ such that all the links of X are connected and we assume that the links of all the vertices of X are finite. Also Γ is a locally compact, properly discontinuous, unimodular group of automorphisms of X acting cocompactly on X .

Following [BS97] we introduce the following notations:

1. For $0 \leq k \leq n$, denote by $\Sigma(k)$ the set of ordered k -simplices (i.e. $\sigma \in \Sigma(k)$ is and ordered $k+1$ -tuple of vertices) and choose a set $\Sigma(k, \Gamma) \subseteq \Sigma(k)$ of representatives of Γ -orbits.
2. For a simplex $\sigma \in \Sigma(k)$, denote by Γ_σ the stabilizer of σ and by $|\Gamma_\sigma|$ the measure of Γ_σ with respect to the Haar measure.

The following proposition is taken from [BS97], [DJ00]:

Proposition 2.1. *[BS97, Lemma 1.3], [DJ00, Lemma 3.3] For $0 \leq l < k \leq n$, let $f = f(\tau, \sigma)$ be a Γ -invariant function on the set of pairs (τ, σ) , where τ is an ordered l -simplex and σ is an ordered k -simplex with $\tau \subset \sigma$. Then*

$$\sum_{\sigma \in \Sigma(k, \Gamma)} \sum_{\substack{\tau \in \Sigma(l) \\ \tau \subset \sigma}} \frac{f(\tau, \sigma)}{|\Gamma_\sigma|} = \sum_{\tau \in \Sigma(l, \Gamma)} \sum_{\substack{\sigma \in \Sigma(k) \\ \tau \subset \sigma}} \frac{f(\tau, \sigma)}{|\Gamma_\tau|}$$

The reader should note, that from now on we will use the above proposition to change the order of summation without mentioning it explicitly.

Definition 2.2. *A weight on X is an equivariant function $m : \bigcup_{k \geq 1} \Sigma(k) \rightarrow \mathbb{R}_{>0}^+$ such that:*

1. For every $\tau = (v_0, \dots, v_k)$ and for every permutation $\sigma \in S_k$ we have $m((v_0, \dots, v_k)) = m((v_{\sigma(0)}, \dots, v_{\sigma(k)}))$.
2. There is a $C(m)$ such that for every $\tau \in \Sigma(1)$ we have the following equality

$$\sum_{\sigma \in \Sigma(2), \tau \subset \sigma} m(\sigma) = 3!C(m)m(\tau)$$

Where $\tau \subset \sigma$ means that all the vertices of τ are contained in σ (with no regard to the ordering).

Example 2.3. In [BS97] the function m was defined as: for every $\tau \in \Sigma(k)$, $m(\tau)$ is the number of (unordered) simplices of dimension n that contain τ . In that case, $C(m) = n - 1$.

Remark 2.4. There is a lot of freedom in our definition of the weight function. Without loss of generality, one can always normalize the weight function such that $C(m) = 1$. It is obvious that in the normalized case the function m is determined by its values on $\Sigma(2)$. We chose not to normalize the weight function in this paper as a matter of convenience and so that the reader could easily compare our results to those proven in [BS97].

Definition 2.5. Let $u \in \Sigma(0)$, denote by X_u the link of u in X , that is, the subcomplex of dimension $n - 1$ consisting on simplices $\sigma = (w_0, \dots, w_k)$ such that $\{u\}, \{w_0, \dots, w_k\}$ are disjoint as sets and $(u, w_0, \dots, w_k) = u\sigma \in \Sigma(k + 1)$. As stated above, X is locally finite which means that X_u is a finite simplicial complex.

1. For $0 \leq k \leq n - 1$, denote by $\Sigma_u(k)$ the set of ordered k -simplices.
2. For a simplex $\sigma \in \Sigma_u(k)$ denote by $m_u(\sigma) = m(u\sigma)$.

2.2 Uniformly convex Busemann non positively curved spaces

In this subsection we will give definitions and some results about uniformly convex Busemann non positively curved spaces.

Let (Z, d) be a unique geodesic complete metric space, i.e. between any two points $x, y \in Z$ there is a unique geodesic connecting x and y . For $x, y \in Z$ and $0 \leq t \leq 1$ denote by $tx + (1 - t)y$ the point on the geodesic connecting x and y such that

$$d(x, tx + (1 - t)y) = td(x, y), d(y, tx + (1 - t)y) = (1 - t)d(x, y)$$

This is of course only a notation because Z need not be a vector space.

Definition 2.6. A uniquely geodesic metric¹ space (Z, d) will be called *uniformly convex* if there is a map $\theta : (0, \infty) \times (0, 2] \rightarrow (0, 1]$ such that for every $r > 0$ and every $\varepsilon \in (0, 2]$, for every three points $a, x, y \in Z$ the following implication holds:

$$\begin{aligned} d(a, x) &\leq r \\ d(a, y) &\leq r \\ d(x, y) &\geq \varepsilon r \end{aligned} \Rightarrow d\left(\frac{1}{2}x + \frac{1}{2}y, a\right) \leq (1 - \theta(r, \varepsilon))r$$

and

$$\forall \varepsilon \in (0, 2], \theta(\varepsilon) = \inf\{\theta(r, \varepsilon) : r > 0\} > 0$$

The map θ is called the *modulus of uniform convexity*.

Remark 2.7. The above definition is taken from [GKM08]. The reader should note there are other (non equivalent) definition for uniformly convex metric space - see for instance [KL10].

Examples of uniformly convex metric spaces:

1. Hilbert spaces.
2. L^p spaces for $1 < p < \infty$.
3. CAT(0) spaces - the modulus of convexity might depend on r but it is bounded by the modulus of convexity of a Hilbert space (for the same ε).

The following proposition is stated and proven in [KL10] (with a slightly different definition of uniform convexity), but we shall repeat the proof here for completeness.

Proposition 2.8. Let (Z, d) be a complete uniquely geodesic uniformly convex metric space, then for closed convex bounded non empty sets $C_n \subset Z$ such that $C_{n+1} \subset C_n$, we have $\bigcap_{n=1}^{\infty} C_n \neq \emptyset$.

Proof. Take arbitrary $x \in Z$. If $x \in \bigcap_{n=1}^{\infty} C_n$ we are done. If not there is some N , such that $d(x, C_N) > 0$ (recall that C_N is closed). Denote $r_n = d(x, C_n)$, then $\{r_n\}$ is an increasing non negative sequence which is bounded from above because C_1 is bounded. Denote $r = \lim r_n \geq r_N > 0$. Define $D_n = C_n \cap \overline{B}(x, r + \frac{1}{n})$, by completeness it is enough to show that $\text{diam}(D_n) \rightarrow 0$, because then $\bigcap_{n=1}^{\infty} C_n \supseteq \bigcap_{n=1}^{\infty} D_n \neq \emptyset$. Assume toward contradiction that $\text{diam}(D_n) \rightarrow d > 0$, then there is some n_0 such that for every $n > n_0$ we have $\frac{1}{n} < \frac{d}{2}$. For every $n > n_0$ we have points $x_n, y_n \in D_n$ such that

$$d(x_n, y_n) > d - \frac{1}{n} > \frac{d}{2} = \frac{d}{2(r + \frac{1}{n})} \left(r + \frac{1}{n}\right) \geq \frac{d}{2(r+1)} \left(r + \frac{1}{n}\right)$$

¹Uniquely geodesic means that every two points has a unique geodesic connecting them. From now on we will assume that our spaces are always uniquely geodesic.

and since $x_n, y_n \in D_n$ we have $d(x, x_n), d(x, y_n) \leq r + \frac{1}{n}$. Note that $\frac{d}{2} \leq r + 1$ because $D_n \subseteq \overline{B}(x, r + 1)$ and therefore $\frac{d}{2(r+1)} \leq 1$. By uniform convexity, for every $n > n_0$ we have

$$\begin{aligned} r_n &\leq d(x, \frac{1}{2}x_n + \frac{1}{2}y_n) \leq \left(1 - \theta(r + \frac{1}{n}, \frac{d}{2(r+1)})\right) \left(r + \frac{1}{n}\right) \leq \\ &\leq \left(1 - \theta\left(\frac{d}{2(r+1)}\right)\right) \left(r + \frac{1}{n}\right) \end{aligned}$$

we can take $n \rightarrow \infty$ and get that

$$r \leq \left(1 - \theta\left(\frac{d}{2(r+1)}\right)\right) r < r$$

which is a contradiction. \square

The above proposition has two useful corollaries:

Corollary 2.9. *Let (Z, d) be a complete uniquely geodesic uniformly convex metric space and let $f : Z \rightarrow \mathbb{R}^+$ be a quasi-convex function, i.e.*

$$\forall 0 \leq t \leq 1, \forall x, y \in Z, f(tx + (1-t)y) \leq \max\{f(x), f(y)\}$$

If there is a $c \in \mathbb{R}^+$ such that the set $\{x : f(x) \leq c\}$ is non empty and bounded, then f has a minimum. Moreover, if f is strictly quasi-convex, i.e.

$$\forall x, y \in Z, x \neq y, f\left(\frac{1}{2}x + \frac{1}{2}y\right) < \max\{f(x), f(y)\}$$

then this minimum is unique.

Proof. Denote $c' = \inf\{f(x) : x \in Z\} \geq 0$, we shall show that c' is the minimum of f . If $c' = c$ we are done because we know that $\{x : f(x) \leq c\}$ is non empty. Otherwise, there is an integer n_0 such that $\frac{1}{n_0} < c - c'$. Then for every $n \geq n_0$, define

$$C_n = \{x : f(x) \leq c' + \frac{1}{n}\}$$

Those are bounded non empty convex sets (because f is quasi-convex) such that $C_{n+1} \subset C_n$ and by the above proposition we get that $\bigcap_{n=n_0}^{\infty} C_n \neq \emptyset$ and for $x \in \bigcap_{n=n_0}^{\infty} C_n$ we get that $f(x) = c'$. If f is strictly quasi-convex then for every two $x, y \in Z$ such that $f(x) = f(y) = c'$ we get that if $x \neq y$ then

$$f\left(\frac{1}{2}x + \frac{1}{2}y\right) < \max\{f(x), f(y)\} = c'$$

which is a contradiction of the definition of c' as the infimum. \square

Definition 2.10. *A uniquely geodesic metric space (Z, d) is said to be non positively curved in the sense of Busemann, if for every three points: $x, y, z \in Z$ one has*

$$d\left(\frac{1}{2}x + \frac{1}{2}z, \frac{1}{2}y + \frac{1}{2}z\right) \leq \frac{1}{2}d(x, y)$$

Observe that the condition stated in the definition above is equivalent to the condition:

$$\forall x, y, x', y' \in Z, d(\frac{1}{2}x + \frac{1}{2}y, \frac{1}{2}x' + \frac{1}{2}y') \leq \frac{1}{2}d(x, y') + \frac{1}{2}d(x', y)$$

Proposition 2.11. *If (Z, d) uniquely geodesic metric space which is non positively curved in the sense of Busemann and uniformly convex, then for every $y \in Z$, the function $d(\cdot, y) : Z \rightarrow \mathbb{R}$ is a convex function.*

Proof. Let $y, x_1, x_2 \in Z$, we need to show that

$$d(\frac{x_1 + x_2}{2}, y) \leq \frac{1}{2}d(x_1, y) + \frac{1}{2}d(x_2, y)$$

Assume WLOG that $d(x_1, y) \geq d(x_2, y)$, then on the geodesic connecting y and x_1 there is a point x'_1 s.t. $d(x'_1, y) = d(x_2, y)$. From uniform convexity we get that

$$d(\frac{x'_1 + x_2}{2}, y) \leq d(x_2, y)$$

(Note this need not be a strict inequality because we might have $x'_1 = x_2$). From the non positive curvature we get that

$$d(\frac{x'_1 + x_2}{2}, \frac{x_1 + x_2}{2}) \leq \frac{1}{2}d(x_1, x'_1) = \frac{1}{2}(d(x_1, y) - d(x_2, y))$$

Therefore

$$\begin{aligned} d(\frac{x_1 + x_2}{2}, y) &\leq d(\frac{x'_1 + x_2}{2}, y) + d(\frac{x'_1 + x_2}{2}, \frac{x_1 + x_2}{2}) \leq \\ &\leq d(x_2, y) + \frac{1}{2}(d(x_1, y) - d(x_2, y)) = \frac{1}{2}d(x_1, y) + \frac{1}{2}d(x_2, y) \end{aligned}$$

□

Last, observe that if (Z, d) is uniquely geodesic, then for every isometry T of Z and for every two points $x, y \in Z$, one has

$$T(\frac{1}{2}x + \frac{1}{2}y) = \frac{1}{2}T(x) + \frac{1}{2}T(y)$$

2.3 Reflexive Banach spaces

In this subsection we will recall some facts about reflexive Banach spaces which will be very similar to the facts we recalled in the previous section.

Proposition 2.12. *Let $(Z, |\cdot|)$ be a reflexive Banach space, then for closed bounded non empty sets $C_n \subset Z$ such that $C_{n+1} \subset C_n$, we have $\bigcap_{n=1}^{\infty} C_n \neq \emptyset$.*

Proof. Every set C_n is closed and bounded and therefore is compact in the weak topology (because Z is reflexive) and therefore $\bigcap_{n=1}^{\infty} C_n \neq \emptyset$. □

As in the previous subsection we get the following corollary (the proof is exactly the same):

Corollary 2.13. *Let $(Z, |\cdot|)$ be a reflexive Banach space and let $f : Z \rightarrow \mathbb{R}^+$ be a quasi-convex function, i.e.*

$$\forall 0 \leq t \leq 1, \forall x, y \in Z, f(tx + (1-t)y) \leq \max\{f(x), f(y)\}$$

If there is a $c \in \mathbb{R}^+$ such that the set $\{x : f(x) \leq c\}$ is non empty and bounded, then f has a minimum. Moreover, if f is strictly quasi-convex, i.e.

$$\forall x, y \in Z, x \neq y, f\left(\frac{1}{2}x + \frac{1}{2}y\right) < \max\{f(x), f(y)\}$$

then this minimum is unique.

Also observe that for every Banach space $(Z, |\cdot|)$ we have

$$\forall x, y, x', y' \in Z, d\left(\frac{1}{2}x + \frac{1}{2}y, \frac{1}{2}x' + \frac{1}{2}y'\right) \leq \frac{1}{2}d(x, y') + \frac{1}{2}d(x', y)$$

(Where d is the usual metric induced by the norm) and that for every $y \in Z$, $d(\cdot, y) : Z \rightarrow \mathbb{R}$ is a convex function.

Last, recall the Mazur-Ulam theorem (see for instance [FJ03]):

Theorem 2.14. *Every surjective isometry between normed spaces is affine.*

Which yield that for every isometry T of $(Z, |\cdot|)$ and for any two points $x, y \in Z$ we have

$$T\left(\frac{1}{2}x + \frac{1}{2}y\right) = \frac{1}{2}T(x) + \frac{1}{2}T(y)$$

2.4 Uniformly convex Busemann non positively curved spaces and Reflexive Banach spaces concluded

In this subsection we conclude the mutual facts gathered in the last two subsections: let (Z, d) be a uniformly convex, uniquely geodesic, Busemann non positively curved, complete metric space or a reflexive Banach space (where d is the metric induced by the norm), then the following holds:

- For every strictly convex function $f : Z \rightarrow \mathbb{R}^+$ if there is a $c \in \mathbb{R}^+$ such that the set $\{x : f(x) \leq c\}$ is non empty and bounded, then f has a unique minimum.
- $$\forall x, y, x', y' \in Z, d\left(\frac{1}{2}x + \frac{1}{2}y, \frac{1}{2}x' + \frac{1}{2}y'\right) \leq \frac{1}{2}d(x, y') + \frac{1}{2}d(x', y)$$
- For every $y \in Z$, $d(\cdot, y) : Z \rightarrow \mathbb{R}$ is a convex function.

- For every isometry T on Z and for every two points $x, y \in Z$, we have

$$T\left(\frac{1}{2}x + \frac{1}{2}y\right) = \frac{1}{2}T(x) + \frac{1}{2}T(y)$$

Note that the summation symbols means two different things: in the uniquely geodesic metric space, $\frac{1}{2}x + \frac{1}{2}y$ means the unique midpoint between x and y and in the Banach case $\frac{1}{2}x + \frac{1}{2}y$ means the average of the two vectors (which is a midpoint, but it need not be unique). From now on we will consider (Z, d) to be either uniformly convex, uniquely geodesic, Busemann non positively curved, complete metric space or a reflexive Banach space and we will use only the mutual facts stated above (and the completeness).

3 Fixed point criteria via links

Let Γ be a group acting on an n -dimensional simplicial complex as above, let (Z, d) be either uniformly convex, uniquely geodesic, Busemann non positively curved, complete metric space or a reflexive Banach space and let $\rho : \Gamma \rightarrow Isom(Z)$. Fix a function $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ with the following properties:

- $f(0) = 0$ and f is strictly monotone increasing.
- f is strictly convex (and therefore $\lim_{x \rightarrow \infty} f(x) = \infty$).
- For every constants $0 \leq \kappa < 1, C \geq 0$ we have $\sum_{k=1}^{\infty} f^{-1}(C\kappa^k) < \infty$

Examples for such functions are $f(x) = x^p$ with $p > 1$ and $f(x) = a_2x^2 + \dots + a_kx^k$ with a_2, \dots, a_k positive.

For every vertex $u \in X$ denote

$$C^0(X_u) = \{\phi : \Sigma_u(0) \rightarrow Z\}$$

and define $E_{u,\phi} : Z \rightarrow \mathbb{R}^+$ as

$$E_{u,\phi}(\xi) = \sum_{v \in \Sigma_u(0)} m((u, v)) f(d(\xi, \phi(v)))$$

Note that $E_{u,\phi}$ is strictly convex, because f is strictly convex and $d(., \phi(v))$ is convex for every v . Since $E_{u,\phi}(\xi) \rightarrow \infty$ as $d(Im(\phi), \xi) \rightarrow \infty$ we get that $E_{u,\phi}$ has a unique minimum. Therefore there is a map $M_u : C^0(X_u(0)) \rightarrow Z$ which send each ϕ to the minimum of $E_{u,\phi}$.

For every $\phi \in C^0(X_u)$ define $d_u\phi : \Sigma_u(1) \rightarrow \mathbb{R}$ to be $d_u\phi(v, w) = d(\phi(v), \phi(w))$.

Define for every vertex $u \in X$ a constant λ_u as following:

$$\lambda_u = \sup\{\lambda : \lambda E_{u,\phi}(M_u\phi) \leq \sum_{\eta \in \Sigma_u(1)} \frac{m_u(\eta)}{2} f(d_u\phi(\eta)), \forall \phi \in C^0(X_u)\}$$

Denote

$$C^0(X, \rho) = \{\phi : \Sigma(0) \rightarrow X : \phi \text{ is equivariant w.r.t } \rho\}$$

Define an operator $M : C^0(X, \rho) \rightarrow C^0(X, \rho)$ as

$$\forall u \in \Sigma(0), M\phi(u) = M_u\phi|_{X_u}$$

Where $\phi|_{X_u}$ is the restriction of ϕ to the link of u .

For $\phi, \phi' \in C^0(X, \rho)$ define $\frac{1}{2}\phi + \frac{1}{2}\phi' \in C^0(X, \rho)$ as

$$\forall u \in \Sigma(0), (\frac{1}{2}\phi + \frac{1}{2}\phi')(u) = \frac{1}{2}\phi(u) + \frac{1}{2}\phi'(u)$$

Define the operator $M' : C^0(X, \rho) \rightarrow C^0(X, \rho)$ as

$$M'\phi = \frac{1}{2}\phi + \frac{1}{2}M\phi$$

Proposition 3.1. *The images of the operators M and M' are indeed contained in $C^0(X, \rho)$.*

Proof. To show that the image of M is contained in $C^0(X, \rho)$, we need to show that for every $\phi \in C^0(X, \rho)$ we have that $M\phi$ is an equivariant map w.r.t ρ , i.e. for every $u \in \Sigma(0)$ and every $\gamma \in \Gamma$ we have that

$$\rho(\gamma).M\phi(u) = M\phi(\gamma.u)$$

Fix some $u \in \Sigma(0)$ and $\gamma \in \Gamma$, then γ take the link of u to the link of $\gamma.u$ and since ϕ is equivariant and the weight m is invariant we get that

$$\begin{aligned} E_{\gamma.u, \phi|_{X_{\gamma.u}}}(\rho(\gamma).\xi) &= \sum_{\gamma.v \in \Sigma_{\gamma.u}(0)} m((\gamma.u, \gamma.v))f(d(\rho(\gamma).\xi, \phi|_{X_{\gamma.u}}(\gamma.v))) = \\ &= \sum_{\gamma.v \in \Sigma_{\gamma.u}(0)} m((u, v))f(d(\rho(\gamma).\xi, \rho(\gamma).\phi|_{X_u}(v))) = \\ &= \sum_{\gamma.v \in \Sigma_{\gamma.u}(0)} m((u, v))f(d(\xi, \phi(v))) = E_{u, \phi}(\xi) \end{aligned}$$

and therefore if $M_u\phi|_{X_u}$ is the unique minimum of $E_{u, \phi|_{X_u}}(\xi)$ then $\rho(\gamma).M_u\phi|_{X_u}$ is the unique minimum of $E_{\gamma.u, \phi|_{X_{\gamma.u}}}(\xi)$ and the map M is equivariant.

To show M' is equivariant we simply recall that for every isometry T of Z and any $x, y \in Z$, we have

$$T(\frac{1}{2}x + \frac{1}{2}y) = \frac{1}{2}T(x) + \frac{1}{2}T(y)$$

□

Define $E(.,.) : C^0(X, \rho) \times C^0(X, \rho) \rightarrow \mathbb{R}$ as

$$E(\phi, \psi) = \sum_{(u,v) \in \Sigma(1, \Gamma)} \frac{m((u,v))}{|\Gamma_{(u,v)}|} f(d(\phi(u), \psi(v)))$$

Proposition 3.2. 1. For every $\phi, \psi \in C^0(X, \rho)$ we have

$$E(\phi, \psi) = \sum_{u \in \Sigma(0, \Gamma)} \frac{1}{|\Gamma_u|} E_{u, \phi}(\psi(u))$$

2. For every $\phi \in C^0(X, \rho)$ we have

$$\frac{C(m)}{2} E(\phi, \phi) = \sum_{u \in \Sigma(0, \Gamma)} \frac{1}{|\Gamma_u|} \sum_{\eta \in \Sigma_u(1)} \frac{m_u(\eta)}{2} f(d_u \phi(\eta))$$

Where $C(m)$ is the constant such that for every $\eta \in \Sigma(1)$ we have

$$\sum_{\sigma \in \Sigma(2), \eta \subset \sigma} m(\sigma) = 3! C(m) m(\eta)$$

3. For every $\phi, \phi', \psi, \psi' \in C^0(X, \rho)$ we have

$$E(\frac{1}{2}\phi + \frac{1}{2}\phi', \frac{1}{2}\psi + \frac{1}{2}\psi') \leq \frac{1}{2}E(\phi, \psi') + \frac{1}{2}E(\phi', \psi)$$

Proof. 1. For every $\phi, \psi \in C^0(X, \rho)$ we have

$$\begin{aligned} & \sum_{u \in \Sigma(0, \Gamma)} \frac{1}{|\Gamma_u|} E_{u, \phi}(\psi(u)) = \\ &= \sum_{u \in \Sigma(0, \Gamma)} \frac{1}{|\Gamma_u|} \sum_{v \in \Sigma_u(0)} m((u, v)) f(d(\psi(u), \phi(v))) = \\ &= \sum_{u \in \Sigma(0, \Gamma)} \frac{1}{|\Gamma_u|} \sum_{\eta \in \Sigma(1), u \subset \eta} \frac{m(\eta)}{2} f(d(\psi(u), \phi(\eta - u))) \end{aligned}$$

Where $\eta - u$ is v for $\eta = (u, v)$ or for $\eta = (v, u)$ (hence the division by 2).

Changing the order of summation gives:

$$\begin{aligned} & \sum_{\eta \in \Sigma(1, \Gamma)} \frac{m(\eta)}{2|\Gamma_\eta|} \sum_{u \in \Sigma(0), u \subset \eta} f(d(\psi(u), \phi(\eta - u))) = \\ &= \sum_{(u,v) \in \Sigma(1, \Gamma)} \frac{m((u,v))}{2|\Gamma_{(u,v)}|} (f(d(\psi(u), \phi(v))) + f(d(\psi(v), \phi(u)))) = \\ &= \sum_{(u,v) \in \Sigma(1, \Gamma)} \frac{m((u,v))}{|\Gamma_{(u,v)}|} f(d(\psi(u), \phi(v))) = E(\phi, \psi) \end{aligned}$$

2. For every $\phi \in C^0(X, \rho)$ we have

$$\begin{aligned} & \sum_{u \in \Sigma(0, \Gamma)} \frac{1}{|\Gamma_u|} \sum_{\eta \in \Sigma_u(1)} \frac{m_u(\eta)}{2} f(d_u \phi(\eta)) = \\ & \sum_{u \in \Sigma(0, \Gamma)} \frac{1}{|\Gamma_u|} \sum_{\sigma \in \Sigma(2), u \subset \sigma} \frac{m(\sigma)}{6} f(d_u \phi(\sigma - u)) \end{aligned}$$

Where again $\sigma - u = (v, w)$ for $\sigma = (u, v, w), (v, u, w), (v, w, u)$. Changing the order of summation gives

$$\begin{aligned} & \sum_{\sigma \in \Sigma(2, \Gamma)} \frac{m(\sigma)}{6|\Gamma_\sigma|} \sum_{u \in \Sigma(0), u \subset \sigma} f(d_u \phi(\sigma - u)) = \\ & \sum_{\sigma \in \Sigma(2, \Gamma)} \frac{m(\sigma)}{6|\Gamma_\sigma|} \sum_{\eta \in \Sigma(1), \eta \subset \sigma} \frac{1}{2} f(d\phi(\eta)) \end{aligned}$$

Where $d\phi((v, w)) = d(\phi(v), \phi(w))$ and the factor $\frac{1}{2}$ is because $(v, w), (w, v) \subset (u, v, w)$. Again we can change the order of summation and get

$$\begin{aligned} & \sum_{\eta \in \Sigma(1, \Gamma)} \frac{f(d\phi(\eta))}{12|\Gamma_\eta|} \sum_{\sigma \in \Sigma(2), \eta \subset \sigma} m(\sigma) = \\ & \sum_{\eta \in \Sigma(1, \Gamma)} \frac{C(m)m(\eta)f(d\phi(\eta))}{2|\Gamma_\eta|} = \frac{C(m)}{2} E(\phi, \phi) \end{aligned}$$

3. For every $\phi, \phi', \psi, \psi' \in C^0(X, \rho)$ we have

$$\begin{aligned} E\left(\frac{1}{2}\phi + \frac{1}{2}\phi', \frac{1}{2}\psi + \frac{1}{2}\psi'\right) &= \sum_{(u, v) \in \Sigma(1, \Gamma)} \frac{m((u, v))}{|\Gamma_{(u, v)}|} f\left(d\left(\frac{1}{2}\phi(u) + \frac{1}{2}\phi'(u), \frac{1}{2}\psi(v) + \frac{1}{2}\psi'(v)\right)\right) \leq \\ &\leq \sum_{(u, v) \in \Sigma(1, \Gamma)} \frac{m((u, v))}{|\Gamma_{(u, v)}|} f\left(\frac{1}{2}d(\phi(u), \psi'(v)) + \frac{1}{2}d(\phi'(u), \psi(v))\right) \end{aligned}$$

Where the inequality follows for properties of Z and for the fact that f is monotone increasing. From convexity of f we get

$$\begin{aligned} & \leq \sum_{(u, v) \in \Sigma(1, \Gamma)} \frac{m((u, v))}{|\Gamma_{(u, v)}|} \left(\frac{1}{2}f(d(\phi(u), \psi'(v))) + \frac{1}{2}f(d(\phi'(u), \psi(v)))\right) = \\ & \frac{1}{2}E(\phi, \psi') + \frac{1}{2}E(\phi', \psi) \end{aligned}$$

□

Denote $\lambda = \min\{\lambda_u : u \in \Sigma(0)\}$, then from the above proposition we get:

Corollary 3.3. 1. For every $\phi \in C^0(X, \rho)$ we have that

$$\frac{C(m)}{2}E(\phi, \phi) \geq \lambda E(M\phi, \phi)$$

2. For every $\phi \in C^0(X, \rho)$ we have that

$$\frac{C(m)}{2}E(\phi, \phi) \geq \lambda E(M'\phi, M'\phi)$$

Proof. 1. For every $\phi \in C^0(X, \rho)$, by 2. in the above proposition we get that

$$\begin{aligned} \frac{C(m)}{2}E(\phi, \phi) &= \sum_{u \in \Sigma(0, \Gamma)} \frac{1}{|\Gamma_u|} \sum_{\eta \in \Sigma_u(1)} \frac{m_u(\eta)}{2} f(d_u \phi(\eta)) \geq \\ &\geq \lambda \sum_{u \in \Sigma(0, \Gamma)} \frac{1}{|\Gamma_u|} E_{u, \phi}(M_u \phi) \geq \lambda E(\phi, M\phi) \end{aligned}$$

Where the first inequality is due to the definition of λ and the second inequality is due to 1. in the above proposition.

2. Due to 3. in the above proposition we get that

$$\begin{aligned} \lambda E(M'\phi, M'\phi) &= \lambda E\left(\frac{1}{2}\phi + \frac{1}{2}M\phi, \frac{1}{2}\phi + \frac{1}{2}M\phi\right) \leq \\ &\leq \frac{1}{2}\lambda E(\phi, M\phi) + \frac{1}{2}\lambda E(M\phi, \phi) \leq \frac{C(m)}{2}E(\phi, \phi) \end{aligned}$$

□

Theorem 3.4. If $\lambda > \frac{C(m)}{2}$ then Γ has a fixed point for every ρ .

Proof. Denote by $\kappa = \frac{C(m)}{2\lambda}$, then $0 \leq \kappa < 1$ and for every $\phi \in C^0(X, \rho)$

$$\kappa E(\phi, \phi) \geq E(M'\phi, M'\phi)$$

Therefore, for every $k \in \mathbb{N}$ we have

$$\kappa^k E(\phi, \phi) \geq E((M')^k \phi, (M')^k \phi)$$

Denote by $\delta = \min\{m(\eta) : \eta \in \Sigma(1)\} > 0$, then for every $(u, v) \in \Sigma(1, \Gamma)$ we have

$$\kappa^k E(\phi, \phi) \geq E((M')^k \phi, (M')^k \phi) \geq \delta f(d((M')^k \phi(u), d((M')^k \phi(v)))$$

and therefore, for every $(u, v) \in \Sigma(1, \Gamma)$ we have

$$f^{-1}\left(\frac{\kappa^k E(\phi, \phi)}{\delta}\right) \geq d((M')^k \phi(u), d((M')^k \phi(v))$$

We also have for every $\phi \in C^0(X, \rho)$ that

$$\kappa E(\phi, \phi) \geq E(M\phi, \phi)$$

and therefore

$$E((M')^k \phi, (M')^k \phi) > \kappa E((M')^k \phi, (M')^k \phi) \geq E((M')^k \phi, M(M')^k \phi)$$

So we have

$$\begin{aligned} E((M')^k \phi, (M')^{k+1} \phi) &= E((M')^k \phi, \frac{1}{2}(M')^k \phi + \frac{1}{2}M(M')^k \phi) \leq \\ &\leq \frac{1}{2}E((M')^k \phi, (M')^k \phi) + \frac{1}{2}E((M')^k \phi, M(M')^k \phi) < E((M')^k \phi, (M')^k \phi) \end{aligned}$$

Therefore, for every $(u, v) \in \Sigma(1, \Gamma)$ we have (as before)

$$f^{-1}\left(\frac{\kappa^k E(\phi, \phi)}{\delta}\right) \geq d((M')^k \phi(u), d((M')^{k+1} \phi(v))$$

By triangle inequality we have for every $u \in \Sigma(0)$

$$2f^{-1}\left(\frac{\kappa^k E(\phi, \phi)}{\delta}\right) \geq d((M')^k \phi(u), d((M')^{k+1} \phi(u))$$

Now since

$$\sum_{k=1}^{\infty} 2f^{-1}\left(\frac{\kappa^k E(\phi, \phi)}{\delta}\right) < \infty$$

Then for every $u \in \Sigma(0)$, $(M')^k \phi(u)$ is Cauchy sequence and therefore we can define $\phi_0 \in C^0(X, \rho)$ as

$$\forall u \in \Sigma(0), \phi_0(u) = \lim(M')^k \phi(u)$$

We get that $E(\phi_0, \phi_0) = 0$ and therefore ϕ_0 must be a constant, equivariant map (so there is a fixed point). \square

Corollary 3.5. *The above theorem generalize several previous theorems:*

1. *For the case $f(x) = x^2$ and Z is a Hilbert space, we get the criterion stated in [BS97] and [Zuk96] (this is the famous Żuk criterion).*
2. *For the case $f(x) = x^2$ and Z is a Hadamard space (i.e. $CAT(0)$ and complete), we get the criterion stated in [IN05] (for the 2 dimensional case).*
3. *For the case $f(x) = x^p$ and Z is L^p for $1 < p < \infty$, we get the criterion in [Bou12] (for the 2 dimensional case).*
4. *For the case $f(x) = x^p$ and Z is a reflexive Banach space, we improve the criterion in [Now].*

Remark 3.6. Note that in the case that Z is a Banach space, the reflexivity of Z and the strict convexity of f were only required to define M uniquely and to insure it is an equivariant operator. We can avoid those restrictions and work in a general Banach space, if we define a different equivariant operator M . For instance, if we define

$$M\phi_u = \sum_{v \in \Sigma_u(0)} \frac{m_u(v)}{\sum_{v \in \Sigma_u(0)} m_u(v)} \phi(v)$$

Then by Mazur-Ulam theorem, M will be an equivariant operator and we have the same criterion with the appropriate λ .

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